## Sections 4.1 - 4.3: The vector spaces R^3, R^n, abstract vector spaces, subspaces, bases, …

In section 4.1 we learn, for R^3:

- what is R^3?
- the arithmetic of vectors
- the axioms for a vector space
- linear dependence and independence
- what is a basis? (A more usual definition is given in 4.4.)
- Subspaces: a criterion for a subset to be a subspace
- the term 'linear combination' is used on p. 218 and there are questions about linear combinations.
- Theorem: 3 vectors in R^3 are independent <=> the matrix they form has det $\neq 0$, and then they are a basis.

Most questions are about linear combinations, dependence and independence.

In section 4.2 we learn, for R^n:

- exactly the same thing.
- In addition, there is a theorem that solutions to a homogeneous system of equations form a subspace.

Most questions are about identifying subspaces and finding bases for solution spaces.

In section 4.3 we learn

- official definition of linear combination
- more emphasis on independent sets of vectors
- spanning sets of vectors
- Theorem: n vectors in R^n are independent <=> the matrix they form has det $\neq 0$, and then they are a basis.

What are R^3 and R^n?

$\mathbb{R}^3$ is the set of triples of real numbers like $\begin{bmatrix} 1 \\ -3 \\ 27 \end{bmatrix}$

The arithmetic of vectors like

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

What is a linear combination of vectors?

It is an expression like

$$2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} + 7\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

Like 4.1 questions 9 - 14, 25 - 28 and also 4.3 questions 9 - 16:
Express the vector t as a linear combination of (1, 0, -1) and (1, 2, -2) or else show that it cannot be done

t = (1, 4, -3), t = (1, 6, -3)

Solution we try to write

$$x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

We solve $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$. Elimination:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ -1 & -2 & -3 & -3 \end{array}\right] \xrightarrow{③ \to ③ + ①} \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & -1 & -2 & -2 \end{array}\right]$$

$$③ \to ③ + \tfrac{1}{2}② \longrightarrow \left[\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution,

$2y = 4$ $y = 2$, $x + y = 1$, $x = -1$

$$\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

It is not possible to write $\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix}$ as a linear combn.

# Pre-class Warm-up!!!

Is it possible to express the vector (2,6) as a linear combination of the vectors (1,3) and (2,5)?

a. Yes ✓ $\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

b. No

Another question:
Is it possible to express the vector (2,5) as a linear combination of the vectors (1,3) and (2,6)?

a. Yes

b. No ✓

Quick approach:
Any linear comb. of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Because we had an exam last week, Quiz 4 tomorrow is on all of 3.1-3.6 (Euler's method 2.4 will not be on the quiz).

## Definitions of linear independence

Definition (page 216) Vectors v_1, … v_s in R^n are linearly dependent if and only if one of them is a linear combination of the others. Otherwise the vectors are linearly independent.

Example: The vectors (1, 0, -1), (1, 2, -2), (1, 4,-3) are dependent because
$$\begin{bmatrix}1\\4\\-3\end{bmatrix} = 2\begin{bmatrix}1\\2\\-2\end{bmatrix} - \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Theorem 3 on page 216: Vectors v_1, … ,v_s are dependent if and only if there exist numbers a_1, … ,a_s, not all 0, with a_1v_1 + … + a_sv_s = 0

Definition/Theorem on page 231: They are independent if and only if the only solution to a_1v_1 + … + a_sv_s = 0 is the zero solution a_1 = a_2 = … = a_s = 0.

Remember these

Question: do you think the vectors (1, 0, 1), (1, 2, -2), (1, 6, -3) are dependent or independent?
a. dependent b. independent c. not sure

Independent we can solve
$$x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}1\\2\\-2\end{bmatrix} + z\begin{bmatrix}1\\6\\-3\end{bmatrix} = 0$$
and $\begin{bmatrix}1&1&1\\0&2&6\\1&-1&-3\end{bmatrix}$ reduces to $\begin{bmatrix}1&?&?\\0&1&?\\0&0&1\end{bmatrix}$

There is a unique solution.

Another example: (1,3), (2,6), (2,5) are linearly dependent, because
$$\begin{bmatrix}2\\6\end{bmatrix} = 2\begin{bmatrix}1\\3\end{bmatrix} + 0\begin{bmatrix}2\\5\end{bmatrix}$$

Thus $2\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0$ verifies the condition in Theorem 3.

Question like 4.1 19-24, and like 4.3 17-22

Determine whether (1, 0, -1), (1, 2, -2), (1, 6, -3) are independent. If not, find a non-zero linear combination of them that equals zero.

Reduce $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ -1 & -2 & -3 \end{bmatrix}$

We get $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$ — independent

$\begin{bmatrix} 1 & 2 & ? \\ 0 & 1 & ? \\ 0 & 0 & 0 \end{bmatrix}$ — dependent

Question:
Are the following sets of vectors dependent or independent?

1. The set (1,3) and (2,4)

a. dependent
b. independent ✓

2. The set (1,3) and (2,6)

a. dependent ✓
b. independent

Also, what about the sets

3. (1,2) and (0,0)? $0\begin{bmatrix}1\\2\end{bmatrix}+17\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

so dependent.

1. (1,3), (2,4), (3,7), (1,0)?

are dependent b/c when we reduce.

$\begin{bmatrix}1 & 2 & 3 & 1\\3 & 4 & 7 & 0\end{bmatrix}$ we get 2 free variables

so infinitely many solutions to

$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}2\\4\end{bmatrix}+z\begin{bmatrix}3\\7\end{bmatrix}+w\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$

# Criterion for independence of n vectors in R^n

Theorem 4 of 4.1 and Theorem 2 of 4.3.
The n vectors v_1, … ,v_n in R^n are linearly independent if and only if the n x n matrix

A = [v_1 v_2 … v_n]

has non-zero determinant.

$\iff$ the matrix $A$ reduces to $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$\iff$ $A$ is invertible,

Example like 4.1 questions 15-18.
Apply Theorem 4 to determine whether the given vectors are dependent or independent.
(1, 0, -1), (1, 2, -2), (1, 6, -3)

$$\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ -1 & -2 & -3 \end{bmatrix} = -6 - 6 + 0 + 2 + 12 + 0 = 2 \neq 0,$$

so independent.

How about (1, 0, -1), (1, 2, -2), (1, 4, -3)?

$\det A = 0$

Another equivalent way to say v_1, … ,v_n are independent.

## Theorem 1 of Sec 4.1 / Axioms for a vector space in Sec 4.2

(a) u+v = v+u
(b) u + (v + w) = (u + v) + w
(c) there is a vector 0 with u + 0 = u = 0 + u always
(d) there is a vector -u with u+(-u) = 0 = (-u) + u always
(e) r(u+v) = ru + rv
(f) (r+s)u = ru + sv
(g) r(su) = (rs)u
(h) 1(u) = u

Examples: 1. $\mathbb{R}^n$ is a vector space

2. The set of all functions $a\sin x + b\cos x + ce^x$ where a, b, c are numbers, is a vector space

3. The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ where $x = 2y$ is a vector space. This is a subspace of $\mathbb{R}^2$

Definition (page 219 and 224)

A subset of a vector space V is called a subspace if it is a vector space in its own right with the given operations of + and scalar multiplication.

Question: which of the following are subspaces of R^2?

a. The set of vectors (x,y) with xy = 0. No
b. The set of vectors (x,y) with 2x + 3y = 0. Yes

a. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not satisfy $1 \cdot 1 = 0$.

b. If $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ satisfy $2x_1 + 3y_1 = 0$ and $2x_2 + 3y_2 = 0$ then $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ satisfies $2(x_1 + x_2) + 3(y_1 + y_2) = 0$ so lies in this set
also $c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ lies in the set.

Theorem 1 of Sec 4.2 also stated at the bottom of page 219 in Sec 4.1.

A non-empty subset W of a vector space V is a subspace of V <=> it satisfies the following two conditions:
(i) If u and v are in W, then so is u + v.
(ii) If u is in W and c is a scalar, then the vector cu is in W.

Proof "$\Leftarrow$" If these two conditions hold then all the axioms follow automatically because W is a subset of a bigger vector space.

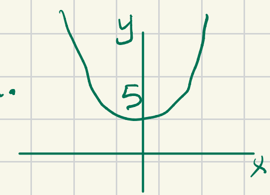
Like Sec 4.1, 29-41 and Sec 4.2, 1-14.
Is the set W of vectors in some R^n a subspace?

a. W is the set of all vectors (x, x^2+5).

b. W is the set of all vectors with x_1 = 4x_3 and x_4 = 5x_2.

c. W is the set of all vectors with x_1x_2 = 0.

d. W is the set of all vectors with $x_1^2+x_2^2=1$.

a.



$\begin{bmatrix}0\\5\end{bmatrix} \in W$ but

$2\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}0\\10\end{bmatrix} \notin W$

so W is not a subspace

# Pre-class Warm-up!!!

Is the set W of vectors (x_1, x_2, x_3, x_4) for which x_1 = 4x_3 and x_4 = 5x_2 a subspace of R^4?

Yes ✓ No

What about the set U of vectors (x_1, x_2) for which

$x_1^2 + x_2^2 = 1$

Is U a subspace of R^2?

Yes No ✓ It is not always the case that $u_1+u_2 \in U$ when $u_1 \in U, u_2 \in U$

e.g. $u_1 = u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $u_1+u_2 \notin U$.

If $x$ and $y$ satisfy $x_1 = 4x_3$, $x_4 = 5x_2$, $y_1 = 4y_3$, $y_4 = 5y_2$ then $x+y$ satisfies $(x_1+y_1) = 4(x_3+y_3)$ and $(x_4+y_4) = 5(x_2+y_2)$

If $c$ is scalar then $cx$ satisfies $(cx_1) = 4(cx_3)$ and $cx_4 = 5(cx_2)$

Can you remember what it means for vectors v_1, v_2, v_3 to be linearly independent?

a. It is possible to write every vector as a linear combination of v_1, v_2, v_3.
b. It is possible to write 0 as a linear combination of v_1, v_2, v_3.
c. There is only one way to write 0 as a linear combination of v_1, v_2, v_3. ✓
d. One of them is a linear combination of the others. Dependent.

Like 4.2 questions 15-22.

Find vectors u and v so that the solution space to the system of equations is the set of all linear combinations su + tv.

x_1 + 2x_2 + 3x_3 = 0
2x_1 + 4x_2 + 6x_3 = 0

Solution. Put the system in matrix form

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \xrightarrow{②\to②-2①}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2$ and $x_3$ are free variables

$x_1 = -2x_2 - 3x_3$

The general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

= the set of all linear combinations of

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Take $u = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Question: Show also that $0u$ is the zero vector $0$ for every vector $u$.

4.2 question 23.
Show that every subspace $W$ of a vector space $V$ contains the zero vector $0$.

4.2 question 27.
Let $u$ and $v$ be fixed vectors in $V$. Show that the set $W$ of all linear combinations $au + bv$ is a subspace of $V$.

Solution. We show that whenever we take two vectors $a_1 u + b_1 v$, $a_2 u + b_2 v$

then $(a_1 u + b_1 v) + (a_2 u + b_2 v) \in W$

This is true because the vector equals

$$(a_1 + a_2) u + (b_1 + b_2) v$$

Also we check $c(a_1 u + b_1 v) \in W$

True because it is $(ca_1)u + (cb_1)v$

Definition in 4.3: the span of vectors v_1, … ,v_k is the set of all linear combinations

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

of these vectors.

It is a subspace of V

We say that v_1, … ,v_k span W if W is the span of $v_1, \ldots, v_k$.

Example:
Find whether the 3 vectors
(1, 0, -1), (1, 2, -2), (1, 6, -3)
a. span R^3
b. are linearly independent.

Solution.

(a) Set up a matrix and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ -1 & -2 & -3 \end{bmatrix}$$

We want to know whether each vector $b \in \mathbb{R}^3$ can be written as a linear combination $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$

We solve $Ax = b$

Reduce $\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 2 & 6 & b_2 \\ -1 & -2 & -3 & b_3 \end{bmatrix}$. We did these numbers before.

and get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array}\right]$

There is always a solution b/c we got the identity on the left. Answer a. Yes.

b.? Yes. There are no zero rows in the echelon form of A.

## Some conditions for independence and spanning not quite expressed this way in the book

Theorem.
Let v_1, … ,v_k be some vectors in R^n, and let A = [v_1 v_2 … v_k] be the matrix with these vectors as the columns.,

a. the vectors are independent <=> the echelon form of A has a leading entry in each column. This implies $k \leq n$.

$\Leftrightarrow$ there are no free variables

$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ 4 vectors in $\mathbb{R}^2 \Rightarrow$ free variables, dependent.

b. the vectors span R^n <=> the echelon form of A has a leading entry in each row (i.e. there are no rows of zeros). This implies $k \geq n$.

$\Leftrightarrow$ we can solve $Ax = b$ for every $b$.

c. if $n = k$, the vectors are independent <=> they span R^n <=> the reduced echelon form is the identity matrix <=> A is invertible <=> $\det A \neq 0$

Question:

1. How long do you think it would take you to determine whether the following vectors are linearly independent in R^3?

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \quad \begin{bmatrix}1\\0\\-1\end{bmatrix} \quad \begin{bmatrix}4\\10\\2\end{bmatrix} \quad \begin{bmatrix}-3\\5\\4\end{bmatrix}$$

a. < 5 seconds ✓ 4 vectors in $\mathbb{R}^3$ are always dependent.
b. between 5 and 20 seconds
c. between 20 seconds and a minute
d. between 1 and 5 minutes
e. can't do it at all

2. How long do you think it would take you to determine whether they span R^3?

Reduce to echelon form. c ?

Question: True or False, for vectors v_1, … , v_6 in R^n ?

a. If v_1, … , v_6 are linearly independent then v_1, … , v_4 are necessarily linearly independent.

b. If v_1, … , v_4 are lin. indep. then v_1, … , v_6 are necessarily lin. indep.

c. b. If v_1, … , v_6 span R^n then v_1, … , v_4 necessarily span R^n.

d. b. If v_1, … , v_4 span R^n then v_1, … , v_6 necessarily span R^n.

True False